



## Technical Note

# A comparison between the enhanced mass transfer in boundary and pressure driven oscillatory flow

Aaron M. Thomas<sup>a,\*</sup>, R. Narayanan<sup>b</sup><sup>a</sup> Department of Chemical Engineering, University of Idaho, P.O. Box 441021, Moscow, ID 83844-1021, USA<sup>b</sup> Department of Chemical Engineering, University of Florida, Gainesville, FL 32611, USA

Received 15 February 2001; received in revised form 21 November 2001

## Abstract

Mass transport in pulsating flow devices using either moving boundaries or oscillating imposed pressure drops are compared with each other by means of a calculation using a simple model. We conclude that there is no difference between the two configurations as long as one is interested only in the power required to move the fluid for the convective mass transport achieved. However, the boundary driven configuration is more efficient if the power is divided by the total mass transport where both the diffusive and convective parts are taken into account, the boundary driven configuration is more efficient. The amplitude of the piston stroke in the pressure driven configuration and the amplitude in the boundary driven case are assumed to be the same, and the inertia of the moving devices themselves are ignored in this calculation. © 2002 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

It is well known [1–4] that the mass transfer of a species is enhanced by several orders of magnitude when it is present in a fluid medium that is subject to oscillatory motion. This enhancement takes place even if there is no net total flow over a cycle of the oscillation. The physics of enhanced mass transfer is explained clearly by Thomas and Narayanan [4].

Now, it turns out that there are several ways to induce oscillatory flow. For example, the tube walls may be oscillated or an oscillating pressure drop may be imposed. In what follows, an analysis is provided to show the differences and to point out the similarities between a configuration driven by an oscillating piston and one by oscillating boundaries. The mass transfer is then calculated and shown to vary between the two cases. Another important parameter that is calculated for each periodic configuration will be the power required to drive the flow. This will then allow a com-

parison of the power required for the mass transfer produced by each method.

## 2. Pressure and boundary driven mass transfer

The physical model is one where a channel of spacing  $h$  and length  $L$  separates two tanks containing a species in a dilute amount in a carrier gas. The concentrations in the tanks are taken to be  $c_2$  and  $c_1$ . In the pressure driven case the oscillatory motion is induced in the fluid in the channel by an oscillating piston with amplitude  $A$ .<sup>1</sup> If  $x$  is the flow direction and  $y$  is transverse to it, then from the equation of motion for a Newtonian fluid in fully developed flow, the axial component of velocity in the simplified case of periodic flow of frequency  $\omega$  in a two-dimensional channel is given by

$$V_x = 2\Re\left(\tilde{V}_x e^{-i\omega t}\right)$$

<sup>1</sup> If the amplitude of oscillation is much less than the length of the channel, end effects can be safely ignored. The effect of secondary flows at the ends of the channel would only lessen the overall effective length.

\* Corresponding author. Tel.: +1-208-885-7652; fax: +1-208-885-7462.

E-mail address: amthomas@uidaho.edu (A.M. Thomas).

### Nomenclature

$A$	peak-to-peak amplitude
$c$	concentration field
$D$	diffusion coefficient (cm <sup>2</sup> /s)
$h$	channel width (cm)
$i$	$\sqrt{-1}$
$L$	length of channel (cm)
$P$	pressure
$\bar{Q}$	time- and space-averaged mass transfer (mol/cm <sup>2</sup> s)
$Sc$	Schmidt number $\frac{v}{D}$
$t$	time (s)
$V_x$	axial velocity (cm/s)
$W$	Womersley number $h\sqrt{\frac{\omega}{2\nu}}$
$x$	axial coordinate
$y$	transverse coordinate
<i>Greek symbols</i>	
$\beta$	constant in equation for velocity

$\eta$	constant in equation for concentration
$\varphi$	constant in equation for concentration
$\mu$	fluid's dynamic viscosity (g/cm s)
$\nu$	fluid kinematic viscosity $\mu/\rho$ (cm <sup>2</sup> /s)
$\omega$	oscillation frequency (rad/s)
$\psi$	constant in equation for velocity

### Superscripts

$\wedge, \sim$	complex conjugates, moving reference
*	dimensionless variable
—	time averaged

### Subscripts

1	left reservoir
2	right reservoir
wall	property at the boundary
conv	convective portion only
total	diffusive and convective portions

where  $\Re$  is the real part of the argument and where

$$\tilde{V}_x = \frac{\frac{1}{4}A\omega\tilde{W}}{(\tilde{W}-2)e^{\tilde{W}} - (\tilde{W}+2)e^{-\tilde{W}} + 4} \times \left[ \left( e^{-\tilde{W}} - 1 \right) e^{\tilde{W}y^*} + \left( 1 - e^{\tilde{W}} \right) e^{-\tilde{W}y^*} + e^{\tilde{W}} - e^{-\tilde{W}} \right] \quad (1)$$

with  $y^* = y/h$  and  $\tilde{W} = (i-1)W$ . The Womersley number  $W$  is defined as  $W \equiv h(\omega/2\nu)^{1/2}$ . Note that the Womersley number has a  $\sqrt{2}$  in the denominator; therefore, it may be different than the definition of the Womersley number of previous authors.

The concentration field  $c$  can be found by solving the species continuity equation and is of the form

$$c = \frac{(c_2 - c_1)x}{L} + 2\Re\left\{ \hat{c}(y)e^{i\omega t} \right\}$$

where

Here,  $\hat{W} = (i+1)W$  and is the conjugate of  $\tilde{W}$  while the Schmidt number is  $Sc \equiv \nu/D$ . The term  $(c_2 - c_1)/L$  is the mean axial concentration gradient. In deriving the total mass transfer,  $\tilde{V}_x$  and  $\hat{c}$  may be re-expressed as

$$\tilde{V}_x = \frac{1}{4}A\omega \left( \psi_1 e^{\tilde{W}y^*} + \psi_2 e^{-\tilde{W}y^*} + \psi_3 \right)$$

$$\hat{c} = \frac{1}{4}A \frac{(c_2 - c_1)}{L} \times \left( \varphi_1 e^{-\hat{W}\sqrt{Sc}y^*} + \varphi_2 e^{\hat{W}\sqrt{Sc}y^*} + \varphi_3 e^{-\hat{W}y^*} + \varphi_4 e^{\hat{W}y^*} + \varphi_5 \right)$$

where the  $\psi$ 's and  $\varphi$ 's are all constants depending on the parameters of the system ( $D$ ,  $\omega$ ,  $\nu$ , and  $h$ ) written this time in terms of the Womersley and Schmidt numbers and where the expressions for the  $\psi$ 's and  $\varphi$ 's can be derived upon inspection of  $\tilde{V}_x$  and  $\hat{c}$  from the above equations. The total time and space averaged mass transfer is defined by

$$\hat{c}x = \frac{\frac{1}{4}i(c_2 - c_1)\hat{W}Sc}{(Sc-1)\left[ (2-\hat{W})e^{\hat{W}} + (2+\hat{W})e^{-\hat{W}} - 4 \right]} \left[ \frac{(1-e^{-\hat{W}})(e^{\hat{W}} - e^{\hat{W}\sqrt{Sc}}) + (e^{\hat{W}} - 1)(e^{\hat{W}\sqrt{Sc}} - e^{-\hat{W}})}{e^{-\hat{W}\sqrt{Sc}} - e^{\hat{W}\sqrt{Sc}}} Sc^{-1/2} e^{-\hat{W}\sqrt{Sc}y^*} \right. \\ + \frac{(1-e^{-\hat{W}})(e^{\hat{W}} - e^{-\hat{W}\sqrt{Sc}}) + (e^{\hat{W}} - 1)(e^{-\hat{W}\sqrt{Sc}} - e^{-\hat{W}})}{e^{-\hat{W}\sqrt{Sc}} - e^{\hat{W}\sqrt{Sc}}} Sc^{-1/2} e^{\hat{W}\sqrt{Sc}y^*} + (e^{\hat{W}} - 1)e^{-\hat{W}y^*} + (1 - e^{-\hat{W}})e^{\hat{W}y^*} \\ \left. + (e^{\hat{W}} - e^{-\hat{W}})(Sc^{-1} - 1) \right] \quad (2)$$

$$\bar{Q} = -\frac{D(c_2 - c_1)}{L} + \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \int_0^1 \tilde{V}_x \hat{c} dy^* dt$$

This then gives

$$\begin{aligned} \bar{Q} = & -\frac{D(c_2 - c_1)}{L} + \frac{A^2\omega(c_2 - c_1)}{8L} \Re \\ & \times \left[ \frac{\psi_1\phi_1 \left( e^{\tilde{W} - \hat{W}\sqrt{Sc}} - 1 \right) - \psi_2\phi_2 \left( e^{\hat{W}\sqrt{Sc} - \tilde{W}} - 1 \right)}{\tilde{W} - \hat{W}\sqrt{Sc}} \right. \\ & + \frac{\psi_1\phi_2 \left( e^{\tilde{W} + \hat{W}\sqrt{Sc}} - 1 \right) - \psi_2\phi_1 \left( e^{-(\tilde{W} + \hat{W}\sqrt{Sc})} - 1 \right)}{\tilde{W} + \hat{W}\sqrt{Sc}} \\ & + \frac{\psi_1\phi_3 \left( e^{\tilde{W} - \hat{W}} - 1 \right) - \psi_2\phi_4 \left( e^{\hat{W} - \tilde{W}} - 1 \right)}{\tilde{W} - \hat{W}} \\ & + \frac{\psi_1\phi_4 \left( e^{\tilde{W} + \hat{W}} - 1 \right) - \psi_2\phi_3 \left( e^{-(\tilde{W} + \hat{W})} - 1 \right)}{\tilde{W} + \hat{W}} \\ & + \frac{\psi_1\phi_5 \left( e^{\tilde{W}} - 1 \right) - \psi_2\phi_5 \left( e^{-\tilde{W}} - 1 \right)}{\tilde{W}} \\ & + \frac{\psi_3\phi_2 \left( e^{\hat{W}\sqrt{Sc}} - 1 \right) - \psi_3\phi_1 \left( e^{\hat{W}\sqrt{Sc}} - 1 \right)}{\hat{W}\sqrt{Sc}} \\ & \left. + \frac{\psi_3\phi_4 \left( e^{\hat{W}} - 1 \right) - \psi_3\phi_3 \left( e^{-\hat{W}} - 1 \right)}{\hat{W}} + \psi_3\phi_5 \right] \end{aligned}$$

The above expression is divided by  $-D(c_2 - c_1)/L$ , which is the average diffusive mass transfer, to give the non-dimensional expression for the total mass transfer. This dimensionless form of the mass transfer is now only dependent upon the Womersley number ( $\tilde{W}$ ), the Schmidt number ( $Sc$ ), and the ratio of the piston amplitude to the distance between the plates ( $A/h$ ). This is a well-known result and similar forms of pressure driven mass transfer can be found in the literature [1–4].

Before proceeding to the analysis of the power required to drive the flow, it is noted that

$$\frac{1}{\mu} \frac{\Delta P}{L} = 2\Re \left( \tilde{P} e^{-i\omega t} \right) \quad (3)$$

and therefore  $\tilde{V}_x$  must be homogeneously dependent upon the pressure term,  $\tilde{P}$ . Likewise, the complex conjugate of  $\tilde{V}_x$ , viz.  $\hat{V}_x$ , depends homogeneously upon  $\hat{P}$ , the conjugate of  $\tilde{P}$ . As the velocity is the inhomogeneous term in the species conservation equation,  $\hat{c}$  and  $\tilde{c}$  must also be homogeneous in  $\hat{P}$  and  $\tilde{P}$  respectively. This will make the convective part of the mass transfer depend quadratically upon pressure, i.e.  $|\tilde{P}|^2$ , and when the pressure drop is induced by oscillating piston heads as described earlier, the convective part must be homogeneous in  $A^2$ . This realization will become important in comparing the pressure driven method to the boundary driven method later on.

Returning now to the piston driven configuration, the power can be found by taking the scalar or dot product of the velocity vector with the equation of motion and then integrating the result over the entire fluid region. This tells us that the energy required to drive the fluid is equal to the kinetic energy plus the frictional heat dissipated by the system. This is only the power required to move the fluid, and it does not account the inertia of the piston itself. For the pressure driven case we have

$$\text{Power} = \frac{\Delta P}{L} \int_0^h V_x dy$$

The time-averaged power,  $\overline{\text{Power}}$ , over one complete cycle is

$$\overline{\text{Power}} = 2\Re \left( \frac{\Delta \tilde{P}}{L} \int_0^h \hat{V}_x dy \right)$$

or

$$\overline{\text{Power}} = \frac{\mu A^2 \omega^2}{8h} \Re \left[ \frac{\tilde{W}^3 \left( e^{\hat{W}} - e^{-\hat{W}} \right)}{\left( 2 - \tilde{W} \right) e^{\hat{W}} + \left( 2 - \tilde{W} \right) e^{-\hat{W}} - 4} \right]$$

Because the geometry is two-dimensional and infinite in extent, the time-averaged power calculated is essentially a power per area of fluid. The dimensionless time-averaged power becomes

$$\frac{\overline{\text{Power}}}{\mu h \omega^2} = \frac{1}{8} \frac{A^2}{h^2} \Re \left[ \frac{\tilde{W}^3 \left( e^{\hat{W}} - e^{-\hat{W}} \right)}{\left( 2 - \tilde{W} \right) e^{\hat{W}} + \left( 2 - \tilde{W} \right) e^{-\hat{W}} - 4} \right] \quad (4)$$

Note from the above equations that the time-averaged power is also a quadratic function of the pressure in much the same way as the convective part of the mass transfer.

Moving on to the case of periodic flow induced by the oscillation of the walls we see that the power and mass transfer for this case are obtained in a manner similar to the pressure driven case. In this configuration, the walls of the channel oscillate in phase as  $V_{\text{wall}} = (1/2) \times A\omega \cos(\omega t)$ . Once again assuming the fluid to be incompressible, Newtonian, with a kinematic viscosity  $\nu$ , and the flow to be laminar,

$$\tilde{V}_x = \frac{1}{4} A\omega \left[ \frac{e^{\tilde{W}y^*} + e^{-\tilde{W}} e^{-\tilde{W}y^*}}{1 + e^{-\tilde{W}}} \right] \quad (5)$$

and

$$\begin{aligned} \hat{c} = & \frac{\frac{1}{4} Ai(c_2 - c_1) \sqrt{Sc} \left( 1 - e^{-\tilde{W}} \right) \left[ e^{-\hat{W}\sqrt{Sc}y^*} + e^{-\hat{W}\sqrt{Sc}} e^{\hat{W}\sqrt{Sc}y^*} \right]}{L(Sc - 1) \left( 1 + e^{-\tilde{W}} \right) \left( e^{-\hat{W}\sqrt{Sc}} - 1 \right)} \\ & + \frac{\frac{1}{4} Ai(c_2 - c_1) Sc}{L(Sc - 1) \left( 1 + e^{-\tilde{W}} \right)} \left[ e^{-\hat{W}} e^{\hat{W}y^*} + e^{-\hat{W}y^*} \right] \quad (6) \end{aligned}$$

for the boundary driven system. The velocity and concentration are related to  $\tilde{V}_x$  and  $\hat{c}$  as before. Again, expressing the velocity and concentration as

$$\tilde{V}_x = \frac{1}{4}A\omega \left( \beta_1 e^{-\tilde{w}y^*} + \beta_2 e^{-\tilde{w}y^*} \right)$$

$$\hat{c} = \frac{\frac{1}{4}A(c_2 - c_1)}{L} \left( \eta_1 e^{-\hat{w}\sqrt{Sc}y^*} + \eta_2 e^{\hat{w}\sqrt{Sc}y^*} + \eta_3 e^{\hat{w}y^*} + \eta_4 e^{-\hat{w}y^*} \right)$$

where the  $\beta$ 's and the  $\eta$ 's can be determined by inspection from Eqs. (5) and (6) respectively, the mass transfer in dimensionless form becomes

$$\frac{\bar{Q}}{\frac{D(c_1 - c_2)}{L}} = 1 - \frac{A^2 W^2 Sc}{4h^2} \Re \left[ \frac{\beta_1 \eta_1 \left( e^{\tilde{w} - \hat{w}\sqrt{Sc}} - 1 \right) - \beta_2 \eta_2 \left( e^{\hat{w}\sqrt{Sc} - \tilde{w}} - 1 \right)}{\tilde{w} - \hat{w}\sqrt{Sc}} + \frac{\beta_1 \eta_2 \left( e^{\tilde{w} + \hat{w}\sqrt{Sc}} - 1 \right) - \beta_2 \eta_1 \left( e^{-(\tilde{w} + \hat{w}\sqrt{Sc})} - 1 \right)}{\tilde{w} + \hat{w}\sqrt{Sc}} + \frac{\beta_1 \eta_3 \left( e^{\tilde{w} + \hat{w}} - 1 \right) - \beta_2 \eta_4 \left( e^{-(\tilde{w} + \hat{w})} - 1 \right)}{\tilde{w} + \hat{w}} + \frac{\beta_1 \eta_4 \left( e^{\tilde{w} - \hat{w}} - 1 \right) - \beta_2 \eta_3 \left( e^{\hat{w} - \tilde{w}} - 1 \right)}{\tilde{w} - \hat{w}} \right]$$

The power required to drive a boundary driven system is once more the sum of the kinetic energy and the fric-

tional heat dissipated by it. This is equal to the product of the velocity and the shear stress at the boundaries, and is therefore

$$\text{Power} = -2\mu V_x \frac{\partial V_x}{\partial y} \Big|_{y=0}$$

After taking the time average over one cycle, the total dimensionless power over a cycle is then

$$\frac{\overline{\text{Power}}}{\mu h \omega^2} = \frac{1}{4} \frac{A^2}{h^2} \Re \left[ \frac{\hat{w} \left( e^{-\hat{w}} - 1 \right)}{1 + e^{-\hat{w}}} \right] \tag{7}$$

Notice that here too, the convective part of the mass transfer and the time-averaged power are both homogeneous in the square of the amplitude of the wall displacement.

### 3. A comparison of the oscillatory methods and some results

Fig. 1 shows the convective mass transfer increasing with Womersley number for each configuration and is a result that is qualitatively similar to that obtained by Harris and Goren [1]. Note that the comparison is made for the same value of  $A$  where  $A$  represents the peak-to-peak amplitude of the piston head in the one case and the peak-to-peak displacement of the channel walls in the other. We see that the convective mass transfer of a single species for both methods under the same conditions and given parameters is greater for the pressure driven method for all Womersley numbers than the boundary driven method. This is conceivably due to the

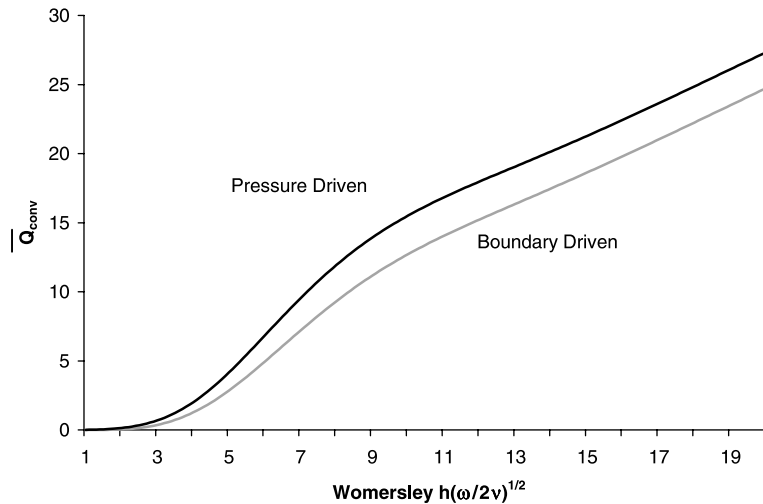


Fig. 1. Convective mass transfer of helium in a nitrogen carrier for a pressure driven and boundary driven configuration,  $Sc = 0.19$  and  $A/h = 10$ .

fact that in a boundary driven configuration, a large fraction of the mass transfer takes place in a small region near the surfaces of the plates where most of the fluid is moving. However, for the pressure driven method, more fluid moves between the two plates due to the imposed pressure drop which gives rise to a larger region for mass transfer to take place. As more fluid moves, the drawback of the pressure driven case is that more power is required to drive it than a boundary driven configuration. The ratio of convective mass transfer to power required for each method therefore gives a reasonable comparison between both systems; the ratio being independent of amplitude in both the pressure and boundary driven cases. This is important because the value of the amplitude of the piston stroke for the pressure driven case is not necessarily the amplitude of the boundary displacement in the boundary driven case. By eliminating the amplitude in the ratio of convective mass transfer to power, the problem of dealing with two different amplitudes is no longer an issue. As seen in Fig. 2(a), it was found that the values for this ratio for all Womersley numbers are identical in both cases. This shows that there must be a direct relationship between the pressure driven and boundary driven configurations. This can be further demonstrated by a moving frame calculation for the boundary driven method.

By making a simple frame change calculation, we can see that the moving boundary case is similar to a pressure driven system with a pressure drop of

$$\frac{\Delta P}{L} = \frac{1}{2} A \omega^2 \cos(\omega t)$$

which is once again dependent on the amplitude of oscillation imposed upon the system. One can imagine sitting on the moving plate and observing the fluid oscillating back and forth. Since the observer cannot determine that the plate is moving, the observer would assume that only a periodic pressure gradient could be causing the fluid to oscillate and this time-dependent “pressure gradient” is observed in the equation of motion for the moving frame. As the power required to drive the system would be the same regardless of the reference point of either observer; the ratio of convective mass transfer to power will be the same for the fixed and moving frame since it is exactly the same system.

Although it may seem surprising that periodic flow driven by an oscillating piston and periodic flow in a boundary driven configuration will give exactly the same convective mass transfer to power ratio, this is also true for any generalized oscillating pressure drop that can be written in the form of Eq. (3).

Next we consider the case where the diffusive mass transfer and the convective mass transfer are both taken into account in the calculation of the total mass transfer. This is important at low frequencies as the diffusive mass transfer, which is independent of the Womersley number, plays an important part in the total mass transfer of the system. The ratio of total mass transfer, which is

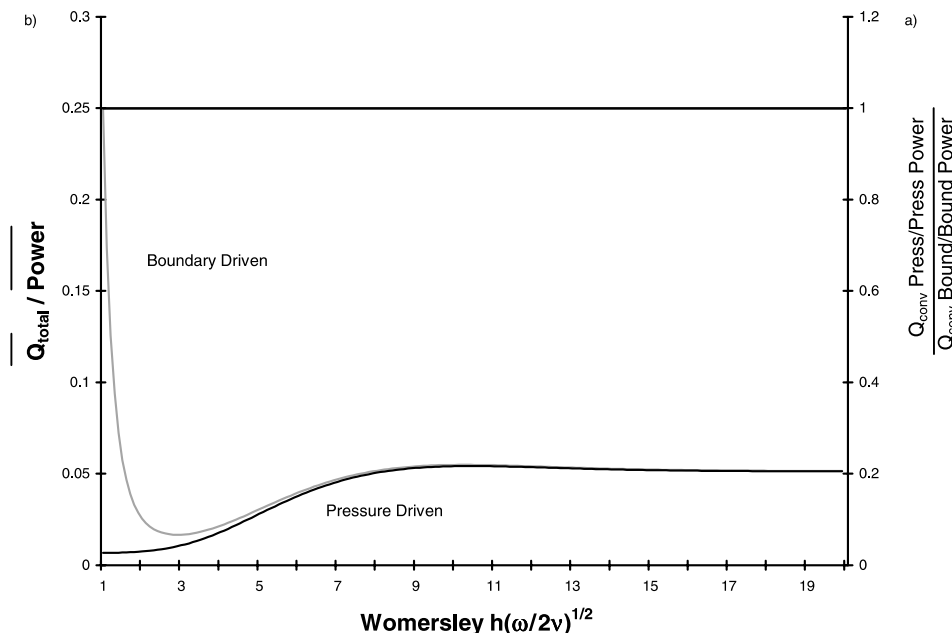


Fig. 2. (a) The ratio of the convective mass transfer per power for the pressure and boundary driven configurations yields unity, right ordinate. (b) The total mass transfer of helium in a nitrogen carrier for a pressure driven and boundary driven configuration,  $Sc = 0.19$  and  $A/h = 10$ , left ordinate.

now the sum of diffusive and convective mass transfer, to the power applied will therefore be different for various Womersley numbers as the power increases for increasing Womersley numbers. If the parameters for the boundary driven and pressure driven cases are the same, the total mass transfer per power applied for the boundary driven case will be higher than the pressure driven case at low Womersley numbers because the power required to drive a pressure driven system is higher than a boundary driven system. This is seen in Fig. 2(b). Of course, the periodic pressure gradient that will give the same ratio of total mass transfer to power as the boundary driven case will be the pressure drop that is derived from the moving frame calculation.

In reality, the power required move the fluid in each configuration could very well be small in comparison to the power requirement due to the inertia of the oscillating piston or moving boundaries. The inertia of the moving parts depends on the density of the materials used and engineering mechanisms required to institute the oscillations. It was not included in this work as it would then be difficult to make a fair comparison between the two systems. However, from a physical and fluid dynamical point of view, comparing the mass transfer of both configurations to the respective power required to move the fluid make sense. This work shows that physically, both configurations are indeed similar, and the same qualitative results are achieved using either method.

#### 4. Conclusions

It has been shown that if the amplitude of a piston stroke and the amplitude of boundary displacement are the same in oscillatory flow then the *convective* mass

transfer for an imposed oscillating pressure drop is greater than the corresponding convective mass transfer in a pulsating boundary system. It has also been conclusively shown that the ratio of convective mass transport to power is the same for both configurations and is independent of the origin of the pressure drop. For the *total* mass transport where the pure molecular diffusion is included, less power is required in the boundary driven configuration at smaller Womersley numbers. At large Womersley numbers, there is little difference between a pressure and boundary driven configuration in the total mass transport. These observations on the fluid mechanics and transport do not take into account possible end effects or the masses associated with the moving parts.

#### Acknowledgements

This work was funded by a NASA Grant no. NAG3 2415 and by an NSF fellowship for Aaron M. Thomas under Grant no. 9616053.

#### References

- [1] H. Harris, S. Goren, Axial diffusion in a cylinder with pulsed flow, *Chem. Eng. Sci.* 22 (1967) 1571–1576.
- [2] E.J. Watson, Diffusion in oscillatory pipe flow, *J. Fluid Mech.* 133 (1983) 233–244.
- [3] U.H. Kurzweg, M.J. Jaeger, Diffusional separation of gases by sinusoidal oscillations, *Phys. Fluids* 30 (1987) 1023–1025.
- [4] A.M. Thomas, R. Narayanan, Physics of oscillatory flow and its effect on the mass transfer and separation of species, *Phys. Fluids* 30 (2001) 859–866.